

Understanding preservation theorems, II

Chaz Schlindwein
Department of Mathematics and Computing
Lander University
Greenwood, South Carolina 29649 USA
`cschlind@lander.edu`

January 6, 2010

Abstract

We present an exposition of much of Sections VI.3 and XVIII.3 from Shelah's book *Proper and Improper Forcing*. This covers numerous preservation theorems for countable support iterations of proper forcing, including preservation of the property "no new random reals over V ," the property "reals of the ground model form a non-meager set," the property "every dense open set contains a dense open set of the ground model," and preservation theorems related to the weak bounding property, the weak ${}^\omega\omega$ -bounding property, and the property "the set of reals of the ground model has positive outer measure."

1 Introduction

This is the fourth of a sequence of papers giving an exposition of portions of Shelah's book, *Proper and Improper Forcing* [9]. The earlier papers were [6], [7], and [8], which cover sections 2 through 8 of [9, Chapter XI], sections 2 and 3 of [9, Chapter XV], and sections 1 and 2 of [9, Chapter VI], respectively.

In this paper, we give an exposition of much of [9, Sections VI.3 and XVIII.3] dealing with preservation theorems. We include proofs of the preservation, under countable support iteration of proper forcing, of the property “no new random reals,” the property “every open dense set contains an old open dense set,” the property of non-meagerness of the reals of the ground model, and preservation theorems related to weak bounding, weak ω -bounding, and “the set of reals of the ground model has positive outer measure.”

Another treatment of preservation theorems, using different methods, is given in [2], [3]. The results of [9, Section VI.3] included here as Theorem 2.5, Theorem 3.5, and Theorem 4.13 may also be derived as corollaries of [1, Theorem 6.1.18]; the proof there is essentially the same as the ones given by Shelah in [9, Section VI.3].

2 Preservation of weak bounding

The most important tool in the study of preservation theorems for countable support forcing iterations is the Proper Iteration Lemma. Here, and throughout this paper, $P_{\alpha,\kappa}$ is characterized by

$$V[G_{P_\alpha}] \models “P_{\alpha,\kappa} = \{p \restriction [\alpha, \kappa) : p \in P_\kappa \text{ and } p \restriction \alpha \in G_{P_\alpha}\}.”$$

Theorem 2.1 (Proper Iteration Lemma, Shelah). *Suppose $\langle P_\eta : \eta \leq \kappa \rangle$ is a countable support forcing iteration based on $\langle Q_\eta : \eta < \kappa \rangle$ and for every $\eta < \kappa$ we have that $\mathbf{1} \Vdash_{P_\eta}$ “ Q_η is proper.” Suppose also that $\alpha < \kappa$ and λ is a sufficiently large regular cardinal and N is a countable elementary submodel of H_λ and $\{P_\kappa, \alpha\} \in N$ and $p \in P_\alpha$ is N -generic and $p \Vdash “q \in P_{\alpha,\kappa} \cap N[G_{P_\alpha}].”$ Then there is $r \in P_\kappa$ such that r is N -generic and $r \restriction \alpha = p$ and $p \Vdash “r \restriction [\alpha, \kappa) \leq q.”$*

Proof: See (e.g.) [8, Theorem 2.1].

We deal first with the weak bounding property.

Definition 2.2. Suppose A and B are sets of integers. We say $A \subseteq^* B$ iff $\{n \in A : n \notin B\}$ is finite.

Definition 2.3. Suppose $\mathcal{P} \subseteq [\omega]^{\aleph_0}$ is a filter. We say \mathcal{P} is a P -filter iff \mathcal{P} contains all co-finite subsets of ω , and $(\forall \mathcal{U} \in [\mathcal{P}]^{\aleph_0})(\exists A \in \mathcal{P})(\forall B \in \mathcal{U})(A \subseteq^* B)$.

Definition 2.4. Suppose \mathcal{P} is a P -filter and P is a forcing notion. We say that P is weakly \mathcal{P} -bounding iff $\mathbf{1} \Vdash_P “(\forall A \in [\omega]^{\aleph_0})(\exists B \in \mathcal{P})(A \not\subseteq^* B)”$.

The following Theorem is [9, Conclusion VI.3.17(1)].

Theorem 2.5. Suppose κ is a limit ordinal and \mathcal{P} is a P -filter and $\langle P_\eta : \eta \leq \kappa \rangle$ is a countable support forcing iteration based on $\langle Q_\eta : \eta < \kappa \rangle$. Suppose for every $\eta < \kappa$ we have P_η is weakly \mathcal{P} -bounding and $\mathbf{1} \Vdash_{P_\eta} “Q_\eta$ is proper.” Then P_κ is weakly \mathcal{P} -bounding.

Proof: This is clear if κ has uncountable cofinality, so assume $\text{cf}(\kappa) = \omega$.

Suppose $p \in P_\kappa$ and A is a P_κ -name and $p \Vdash “A \in [\omega]^{\aleph_0}”$. Let λ be a sufficiently large regular cardinal and N a countable elementary substructure of H_λ such that $\{P_\kappa, \mathcal{P}, A, p\} \in N$.

Let $\langle \alpha_k : k \in \omega \rangle \in N$ be an increasing sequence cofinal in κ such that $\alpha_0 = 0$. Fix $B \in \mathcal{P}$ such that $(\forall X \in \mathcal{P} \cap N)(B \subseteq^* X)$. It suffices to show $p \not\Vdash “A \subseteq^* B”$.

Build $\langle q_k, p_k, m_k : k \in \omega \rangle$ such that $q_0 = p$ and for every $k \in \omega$ we have that each of the following holds:

- (1) $p_k \in P_{\alpha_k}$ is N -generic, and
- (2) $p_k \Vdash “q_{k+1} \in P_{\alpha_k, \kappa} \cap N[G_{P_{\alpha_k}}]$ and $q_{k+1} \leq q_k \restriction [\alpha_k, \kappa)”$ and
- (3) m_k is a P_{α_k} -name for an integer and $p_k \Vdash “\text{if } k > 0 \text{ then } m_k > m_{k-1},”$ and
- (4) $p_k \Vdash “q_{k+1} \Vdash ‘m_k \notin B \text{ and } m_k \in A,’”$ and
- (5) $p_{k+1} \restriction \alpha_k = p_k$, and
- (6) $p_k \Vdash “p_{k+1} \restriction [\alpha_k, \alpha_{k+1}] \leq q_{k+1} \restriction \alpha_{k+1}.”$

The construction proceeds as follows. Given p_k, q_k , and m_{k-1} , work in $V[G_{P_{\alpha_k}}]$ with $p_k \in G_{P_{\alpha_k}}$.

Build $A_k \in [\omega]^{\aleph_0} \cap N[G_{P_{\alpha_k}}]$ and $\langle q_k^i : i \in \omega \rangle \in N[G_{P_{\alpha_k}}]$ such that $q_k^0 = q_k \restriction [\alpha_k, \kappa)$, and for every $i \in \omega$ we have $q_k^{i+1} \leq q_k^i$, and $q_k^{i+1} \Vdash “i \in A \text{ iff } i \in A_k.”$

Using the hypothesis on P_{α_k} we may choose $B_k \in \mathcal{P}$ such that $A_k \not\subseteq^* B_k$. By elementarity we may assume $B_k \in N[G_{P_{\alpha_k}}]$. Because p_k is N -generic, we have $B_k \in V \cap N[G_{P_{\alpha_k}}] = N$.

Because $B_k \in N$ we have $B \subseteq^* B_k$, and hence $A_k \not\subseteq^* B$. Therefore we may choose $m_k \in \omega$ such that $m_k \in A_k$ and $m_k \notin B$ and if $k > 0$ then $m_k > m_{k-1}$.

Let $q_{k+1} = q_k^{m_k+1}$. Clearly (2), (3), and (4) are satisfied.

Using the Proper Iteration Lemma we may choose p_{k+1} satisfying (1), (5), and (6).

This completes the recursive construction.

Let $r \in P_\kappa$ be such that for every $k \in \omega$ we have $r \restriction \alpha_k = p_k$.

Suppose, towards a contradiction, that $r' \leq r$ and $r' \Vdash "A \subseteq^* B."$ Fix P_κ -names n and k such that $r' \Vdash "A \subseteq n \cup B,$ and $m_k > n."$

By strengthening r' we may assume that k and m_k are integers rather than merely names.

Because $r' \leq (p_{k+1}, q_{k+1})$ we have $r' \Vdash "m_k \in A - n \subseteq B,$ and $m_k \notin B."$ This is a contradiction.

The Theorem is established.

Lemma 2.8. *Suppose P is weakly \mathcal{P} -bounding and $\mathbf{1} \Vdash_P "Q$ is almost \mathcal{P} -bounding." Then $P * Q$ is weakly \mathcal{P} -bounding.*

Proof: Suppose $(p, q) \Vdash_{P*Q} "A \in [\omega]^{\aleph_0}."$ Take q' and B in V^P such that $p \Vdash "B \in \mathcal{P}$ and $q' \leq q$ and

(*) $q' \Vdash "(\forall Y \in [\omega]^{\aleph_0} \cap V[G_P])(A \cap Y \not\subseteq^* B)."$

Take $p' \leq p$ and $B' \in \mathcal{P}$ such that $p' \Vdash "B \not\subseteq^* B'."$ By (*) we have $(p', q') \Vdash "A \cap (B - B') \not\subseteq^* B."$ Hence $(p', q') \Vdash "A \not\subseteq^* B'."$

The Lemma is established.

Theorem 2.9. *Suppose \mathcal{P} is a P -filter and $\langle P_\eta : \eta \leq \kappa \rangle$ is a countable support forcing iteration based on $\langle Q_\eta : \eta < \kappa \rangle$. Suppose for every $\eta < \kappa$ we have $\mathbf{1} \Vdash_{P_\eta} "Q_\eta$ is proper and almost \mathcal{P} -bounding." Then P_κ is weakly \mathcal{P} -bounding.*

Proof: By Theorem 2.5 and Lemma 2.8.

3 Preservation of weakly ${}^\omega\omega$ -bounding

In this section we give an exposition of a preservation theorem, due to Shelah, concerning the weak ${}^\omega\omega$ -bounding property.

Definition 3.1. *Suppose f and g are in ${}^\omega\omega$. We say $f \leq^* g$ iff $(\exists n \in \omega)(\forall k > n)(f(k) \leq g(k))$.*

Definition 3.2. Suppose $F \subseteq {}^\omega\omega$ and $g \in {}^\omega\omega$. We say that g bounds F iff $(\forall f \in F)(f \leq^* g)$.

Definition 3.3. Suppose P is a forcing notion. We say that P is weakly ${}^\omega\omega$ -bounding iff $\mathbf{1} \Vdash “(\forall f \in {}^\omega\omega)(\exists g \in {}^\omega\omega \cap V)(g \not\leq^* f)”$.

The following Theorem is [9, Conclusion VI.3.17(2)].

Theorem 3.4. Suppose κ is a limit ordinal and $\langle P_\eta : \eta \leq \kappa \rangle$ is a countable support forcing iteration based on $\langle Q_\eta : \eta < \kappa \rangle$. Suppose for every $\eta < \kappa$ we have P_η is weakly ${}^\omega\omega$ -bounding and $\mathbf{1} \Vdash_{P_\eta} “Q_\eta$ is proper.” Then P_κ is weakly ${}^\omega\omega$ -bounding.

Proof: Use the proof of Theorem 2.5 with $([\omega]^{\aleph_0}, \mathcal{P}, \supseteq^*)$ replaced with $({}^\omega\omega, {}^\omega\omega \cap V, \leq^*)$.

The Theorem is established.

The following definition is equivalent to [9, Definition VI.3.5(1)].

Definition 3.5. Suppose P is a forcing notion. We say P is almost ${}^\omega\omega$ -bounding iff $\mathbf{1} \Vdash “(\forall f \in {}^\omega\omega)(\exists g \in {}^\omega\omega \cap V)(\forall A \in [\omega]^{\aleph_0} \cap V)(\exists^\infty n \in A)(f(n) < g(n))”$.

Lemma 3.6. Suppose P is almost ${}^\omega\omega$ -bounding. Then P is weakly ${}^\omega\omega$ -bounding.

Proof: Take $A = \omega$ in Definition 3.5.

Lemma 3.7. Suppose P is weakly ${}^\omega\omega$ -bounding and $\mathbf{1} \Vdash_P “Q$ is almost ${}^\omega\omega$ -bounding.” Then $P * Q$ is weakly ${}^\omega\omega$ -bounding.

Proof: Like Lemma 2.7.

The Lemma is established.

Theorem 3.9. Suppose $\langle P_\eta : \eta \leq \kappa \rangle$ is a countable support forcing iteration based on $\langle Q_\eta : \eta < \kappa \rangle$. Suppose for every $\eta < \kappa$ we have $\mathbf{1} \Vdash_{P_\eta} “Q_\eta$ is proper and almost ${}^\omega\omega$ -bounding.” Then P_κ is weakly ${}^\omega\omega$ -bounding.

Proof: By Theorem 3.4 and Lemma 3.7.

4 Preservation of no new random reals

We now turn our attention to the preservation of the property “no new random reals.”

Definition 4.1. For $\tau \in {}^{<\omega}2$, we let $U_\tau = \{\eta \in {}^\omega 2 : \eta \text{ extends } \tau\}$.

Recall that for $A \subseteq {}^\omega 2$, the outer measure of A is $\mu^*(A) = \inf\{\Sigma\{2^{-\text{lh}(\tau)} : \tau \in C\} : C \subseteq {}^{<\omega}2 \text{ and } A \subseteq \bigcup\{U_\tau : \tau \in C\}\}$. A is Lebesgue measurable iff $(\forall \tau \in {}^{<\omega}2)$ $(\mu^*(A \cap U_\tau) + \mu^*(U_\tau - A) = \mu^*(U_\tau))$, in which case we write $\mu(A) = \mu^*(A)$.

Definition 4.2. Suppose $A \subseteq {}^\omega 2$. We say that A is closed under rational translation iff $(\forall b \in A)(\forall b^* =_{\text{a.e.}} b)(b^* \in A)$.

The following Lemma is known as “Kolmogorov’s zero-one Law.”

Lemma 4.3. Suppose $A \subseteq {}^\omega 2$ is closed under rational translations and suppose that A is Lebesgue measurable. Then $\mu(A) = 0$ or $\mu(A) = 1$.

Proof: Let $\gamma = \mu(A)$ and suppose, towards a contradiction, that $0 < \gamma < 1$.

Claim 1. Whenever $\tau \in {}^{<\omega}2$ and τ_0 and τ_1 are the immediate successors of τ , then $\mu(A \cap U_{\tau_0}) = \mu(A \cap U_{\tau_1})$.

Proof: We have $\mu(A \cap U_{\tau_0}) = 2^{-\text{lh}(\tau_0)}\mu(\{b \in {}^\omega 2 : \tau_0 \hat{\ } b \in A\}) = 2^{-\text{lh}(\tau_1)}\mu(\{b \in {}^\omega 2 : \tau_1 \hat{\ } b \in A\}) = \mu(A \cap U_{\tau_1})$.

Claim 2: For all $\tau \in {}^{<\omega}2$ we have $\mu(A \cap U_\tau) = 2^{-\text{lh}(\tau)}\gamma$.

Proof: By induction on τ , using Claim 1.

Choose $\delta > \gamma$ such that $\delta^2 < \gamma$. Choose $C \subseteq {}^{<\omega}2$ such that $A \subseteq \bigcup\{U_\tau : \tau \in C\}$ and $\Sigma\{\mu(U_\tau) : \tau \in C\} < \delta$.

For each $\tau \in C$, we may, using Claim 2, choose $C_\tau \subseteq {}^{<\omega}2$ such that $A \cap U_\tau \subseteq \bigcup\{U_\eta : \eta \in C_\tau\}$ and $\Sigma\{\mu(U_\eta) : \eta \in C_\tau\} < 2^{-\text{lh}(\tau)}\delta$.

Let $C^* = \bigcup\{C_\tau : \tau \in C\}$.

We have that $A \subseteq \bigcup\{U_\eta : \eta \in C^*\}$ and $\Sigma\{\mu(U_\eta) : \eta \in C^*\} < \delta^2 < \gamma$.

This contradiction establishes the Lemma.

Definition 4.4. Suppose $Y \subseteq {}^\omega 2$. We define $\text{RT}(Y)$, the “rational translates” of Y , to equal $\{b \in {}^\omega 2 : (\exists b' \in Y)(b' =_{\text{a.e.}} b)\}$.

Definition 4.5. Suppose y and y' are perfect subsets of ${}^\omega 2$ of positive Lebesgue measure. We define $y \preceq y'$ to mean $y \subseteq \text{RT}(y')$.

Lemma 4.6. Suppose $\langle y_n : n \in \omega \rangle$ is a sequence of perfect subsets of ${}^\omega 2$ of positive Lebesgue measure. Then there is a perfect set $y \subseteq {}^\omega 2$ of positive Lebesgue measure such that $(\forall n \in \omega)(y \preceq y_n)$.

Proof: By Lemma 4.3 we have that $\mu(\text{RT}(y_n)) = 1$ for every $n \in \omega$. For each $n \in \omega$ let $D_n \subseteq {}^\omega 2$ be an open set such that $\mu(D_n) < 2^{-n-1}$ and $D_n \cup$

$\text{RT}(y_n) = {}^\omega 2$. Let $C = {}^\omega 2 - \bigcup \{D_n : n \in \omega\}$. We have that C is a closed set of positive measure. Let y be the perfect kernel of C (see [5, page 66]). We have that y is a perfect set of positive measure, and for every $n \in \omega$ we have $y \subseteq C \subseteq {}^\omega 2 - D_n \subseteq \text{RT}(y_n)$.

The Lemma is established.

Lemma 4.7. *Suppose x and y are subsets of ${}^\omega 2$. Then $x \cap \text{RT}(y) = \emptyset$ iff $\text{RT}(x) \cap y = \emptyset$.*

Proof: Clear.

Lemma 4.8. *Let P be any forcing. Then $V[G_P] \models “(\forall x \in {}^\omega 2)(x \text{ is random over } V \text{ iff } (\forall y \in V)(y \text{ is a perfect set of positive Lebesgue measure implies } x \in \text{RT}(y)))”$.*

Proof: Work in $V[G_P]$. Suppose $x \in {}^\omega 2$ is not random over V . Let $B \in V$ be a Borel set such that $x \in B$ and $\mu(B) = 0$. Let $D \in V$ be an open set such that $\mu(D) < 1$ and $\text{RT}(B) \subseteq D$. Let y be the perfect kernel of ${}^\omega 2 - D$. Then $y \in V$ is a perfect set of positive measure, and because $y \cap \text{RT}(B) = \emptyset$, we have $\text{RT}(y) \cap B = \emptyset$, and therefore $x \notin \text{RT}(y)$.

In the other direction, suppose $x \in {}^\omega 2$ and $y \in V$ is a perfect set of positive measure such that $x \notin \text{RT}(y)$. We show that x is not random over V . Choose $\langle D_n : n \in \omega \rangle \in V$ a sequence of open sets such that for every $n \in \omega$ we have $\mu(D_n) < 1/n$ and ${}^\omega 2 - \text{RT}(y) \subseteq D_n$. Let $B = \bigcap \{D_n : n \in \omega\}$. We have that $B \in V$ is a Borel set of Lebesgue measure zero and $x \in B$. Therefore x is not random over V .

The Lemma is established.

The following is [9, Lemma VI.3.18]. Notice how the argument parallels the proof of Theorem 2.5.

Theorem 4.9. *Suppose κ is a limit ordinal and $\langle P_\eta : \eta \leq \kappa \rangle$ is a countable support forcing iteration based on $\langle Q_\eta : \eta < \kappa \rangle$. Suppose for every $\eta < \kappa$ we have $\mathbf{1} \Vdash_{P_\eta} “Q_\eta \text{ is proper and there are no reals that are random over } V.”$ Then $\mathbf{1} \Vdash_{P_\kappa} “\text{there are no reals that are random over } V.”$*

Proof: For $\text{cf}(\kappa) > \omega$ this is clear, so assume instead that $\text{cf}(\kappa) = \omega$.

Suppose $p \in P_\kappa$ and g is a P_κ -name and $p \Vdash “g \in {}^\omega 2.”$ Let λ be a sufficiently large regular cardinal and let N be a countable elementary substructure of H_λ

containing $\{P_\kappa, p, g\}$. Let $\langle \alpha_n : n \in \omega \rangle \in N$ be an increasing sequence cofinal in κ such that $\alpha_0 = 0$.

Using Lemma 4.6, fix $y \subseteq {}^\omega 2$ a perfect set of positive Lebesgue measure such that for every perfect $y' \in N$ with $\mu(y') > 0$ we have $y \preceq y'$.

Build $\langle q_k, p_k, m_k : k \in \omega \rangle$ such that $q_0 = p$ and for each $k \in \omega$ we have the following:

- (1) $p_k \in P_{\alpha_k}$ is N -generic, and
- (2) $p_{k+1} \restriction \alpha_k = p_k$, and
- (3) $p_k \Vdash "q_{k+1} \in P_{\alpha_k, \kappa} \cap N[G_{P_{\alpha_k}}] \text{ and } q_{k+1} \leq q_k \restriction [\alpha_k, \kappa]"$ and
- (4) $p_k \Vdash "p_{k+1} \restriction [\alpha_k, \alpha_{k+1}] \leq q_{k+1} \restriction \alpha_{k+1}"$ and
- (5) $p_k \Vdash "m_{k+1} > m_k \text{ and } q_{k+1} \Vdash (\forall \rho \in {}^{m_k} 2)(U_{\rho \restriction g \restriction [m_k, m_{k+1}]} \cap y = \emptyset)"$

The construction proceeds as follows. Suppose we are given p_k and q_k and m_k . Work in $V[G_{P_{\alpha_k}}]$ with $p_k \in G_{P_{\alpha_k}}$. Build $\langle q_k^i : i \in \omega \rangle \in N[G_{P_{\alpha_k}}]$ a decreasing sequence of conditions in $P_{\alpha_k, \kappa}$ and $f_k \in {}^\omega 2 \cap N[G_{P_{\alpha_k}}]$ such that $q_k^0 \leq q_k$ and for every $i \in \omega$ we have $q_k^i \Vdash "f_k(i) = g(i)"$. Using the hypothesis on P_{α_k} and Lemma 4.8, we may choose a perfect set $y_k \in V$ of positive measure such that $f_k \notin \text{RT}(y_k)$. By elementarity we may assume $y_k \in N[G_{P_{\alpha_k}}] \cap V = N$.

Because $y \preceq y_k$ we have $\text{RT}(y) \subseteq \text{RT}(y_k)$, and hence $f_k \notin \text{RT}(y)$. Hence by Lemma 4.7 we have $\text{RT}(\{f_k\}) \cap y = \emptyset$. Hence for each $\rho \in {}^{m_k} 2$, we may let m_k^ρ be an integer greater than m_k such that $U_{\rho \restriction f_k \restriction [lh(\rho), m_k^\rho]} \cap y = \emptyset$, using the fact that y is closed.

Let $m_{k+1} = \max\{m_k^\rho : \rho \in {}^{m_k} 2\}$. Let $q_{k+1} = q_k^{m_{k+1}+1}$. We have that q_{k+1} satisfies (3) and (5). Using the Proper Iteration Lemma, we may choose p_{k+1} satisfying (1), (2), and (4).

This completes the recursive construction.

Let $r \in P_\kappa$ be chosen such that $(\forall k \in \omega)(r \restriction \alpha_k = p_k)$.

We have $r \Vdash " \text{RT}(\{g\}) \cap y = \emptyset "$. Hence by Lemmas 4.7 and 4.8 we have $r \Vdash "g \text{ is not random over } V."$

The Theorem is established.

5 Preservation of “every new dense open set contains an old dense open set”

In this section we prove preservation of the property “every new dense open set contains an old dense open set.” Shelah includes two very different proofs of this

fact in his book; we follow the proof given in [9, Section XVIII.3].

Throughout this section we fix an enumeration $\langle \eta_n^* : n \in \omega \rangle$ of ${}^{<\omega}\omega$ such that whenever η_i^* is an initial segment of η_j^* then $i \leq j$. Also, throughout this section we let \mathcal{B} equal the set of functions from ${}^{<\omega}\omega$ into ${}^{<\omega}\omega$.

Definition 5.1. Suppose f and g are in \mathcal{B} . We say $f \leq_{\mathcal{B}} g$ iff for every $\eta \in {}^{<\omega}\omega$ there is $\nu \in {}^{<\omega}\omega$ such that $\nu \hat{\ } f(\nu)$ is an initial segment of $\eta \hat{\ } g(\eta)$.

We remark that Definition 5.1 differs from [9, Context and Definition XVIII.3.7A] because we have incorporated [9, Remark XVIII.3.7F(1)].

Lemma 5.2. The relation $\leq_{\mathcal{B}}$ is a partial ordering of \mathcal{B} .

Proof: Immediate.

Lemma 5.3. Suppose $\langle f_i : i \in \omega \rangle$ is a sequence of elements of \mathcal{B} . Then there is $g \in \mathcal{B}$ such that for every $i \in \omega$ we have $f_i \leq_{\mathcal{B}} g$.

Proof: For every $\eta \in {}^{<\omega}\omega$ and $k \in \omega$ define $g_0(\eta) = \eta$ and $g_{k+1}(\eta) = g_k(\eta) \hat{\ } f_k(\eta \hat{\ } g_k(\eta))$. Define $g(\eta)$ to equal $g_n(\eta)$ where $\eta = \eta_n^*$.

To see that $f_k \leq_{\mathcal{B}} g$ it suffices to note that whenever $n > k$ then

$$(\eta_n^* \hat{\ } g_k(\eta_n^*)) \hat{\ } f_k(\eta_n^* \hat{\ } g_k(\eta_n^*)) = \eta_n^* \hat{\ } g_{k+1}(\eta_n^*) \subseteq \eta_n^* \hat{\ } g(\eta_n^*).$$

The Lemma is established.

Lemma 5.4. Suppose P is a forcing notion. Then every dense open subset of ${}^\omega\omega$ in $V[G_P]$ contains a dense open subset of ${}^\omega\omega$ in V iff $V[G_P] \models “(\forall f \in \mathcal{B})(\exists g \in \mathcal{B} \cap V)(f \leq_{\mathcal{B}} g)”$.

Proof: We first establish the “if” direction. Work in $V[G_P]$. Suppose D is a dense open subset of ${}^\omega\omega$. Pick $f \in \mathcal{B}$ such that for every $\eta \in {}^{<\omega}\omega$ we have $U_{\eta \hat{\ } f(\eta)} \subseteq D$. Fix $g \in \mathcal{B} \cap V$ such that $f \leq_{\mathcal{B}} g$. Let $D' = \bigcup \{U_{\eta \hat{\ } g(\eta)} : \eta \in {}^{<\omega}\omega\}$. We have D' is a dense open subset of D and $D' \in V$.

For the “only if” direction, suppose $f \in \mathcal{B}$. Build $\langle D_n, \eta_n, x_n : n \in \omega \rangle$ recursively such that for every $n \in \omega$ we have that either $U_{\eta_n^*} \subseteq \bigcup \{D_i : i < n\}$ and $D_n = D_{n-1}$ and $x_n = x_{n-1}$ and $\eta_n = \eta_{n-1}$, or all of the following:

- (1) η_n extends η_n^* , and
- (2) $D_n = U_{\eta_n \hat{\ } f(\eta_n) \hat{\ } \langle 0 \rangle}$, and
- (3) D_n is disjoint from $\bigcup \{D_i : i < n\} \cup \{x_i : i < n\}$, and
- (4) $x_n \in {}^\omega\omega$ extends $\eta_n \hat{\ } f(\eta_n) \hat{\ } \langle 1 \rangle$.

We may take $D' \in V$ open dense such that $D' \subseteq \bigcup \{D_n : n \in \omega\}$.

Choose $g \in \mathcal{B} \cap V$ such that $(\forall \eta \in {}^{<\omega}\omega)(U_{\eta \hat{\wedge} g(\eta)} \subseteq D')$.

Given $\eta \in {}^{<\omega}\omega$, pick $n \in \omega$ such that $U_{\eta \hat{\wedge} g(\eta)} \cap U_{\eta_n \hat{\wedge} f(\eta_n) \hat{\wedge} \langle 0 \rangle} \neq \emptyset$. We have $x_n \notin D'$, and so $U_{\eta \hat{\wedge} g(\eta)} \subseteq U_{\eta_n \hat{\wedge} f(\eta_n)}$. It follows that $f \leq_{\mathcal{B}} g$.

The Lemma is established.

The following is [9, Conclusion VI.2.15D] and [9, Claim XVIII.3.7D]; we follow the proof given in [9, Chapter XVIII].

Theorem 5.5. *Suppose $\langle P_\eta : \eta \leq \kappa \rangle$ is a countable support forcing iteration based on $\langle Q_\eta : \eta < \kappa \rangle$. Suppose for every $\eta < \kappa$ we have $\mathbf{1} \Vdash_{P_\eta}$ “ Q_η is proper and $\mathbf{1} \Vdash_{Q_\eta}$ ‘for every dense open $D \subseteq {}^\omega\omega$ there is a dense open $D' \subseteq D$ such that $D' \in V[G_{P_\eta}]$.’” Then $\mathbf{1} \Vdash_{P_\kappa}$ “for every dense open $D \subseteq {}^\omega\omega$ there is a dense open $D' \subseteq D$ such that $D' \in V$.’”*

Proof: By induction on κ . The induction step is clear for κ a successor ordinal and, in light of Lemma 5.4, it is likewise clear for κ of uncountable cofinality. So we assume $\text{cf}(\kappa) = \omega$.

Suppose $p \in P_\kappa$ and f is a P_κ -name and $p \Vdash$ “ $f \in \mathcal{B}$.’” Choose λ a sufficiently large regular cardinal and N a countable elementary substructure of H_λ such that $\{P_\kappa, f, p\} \in N$.

Let $\langle \alpha_k : k \in \omega \rangle \in N$ be an increasing sequence cofinal in κ such that $\alpha_0 = 0$. Using Lemma 5.3, fix $g \in \mathcal{B}$ such that $(\forall h \in \mathcal{B} \cap N)(h \leq_{\mathcal{B}} g)$.

Build $\langle q_k, p_k, m_k : k \in \omega \rangle$ such that $q_0 = p$ and for every $k \in \omega$ we have that each of the following holds:

- (1) $p_k \in P_{\alpha_k}$ is N -generic, and
- (2) $p_k \Vdash$ “ $q_{k+1} \in P_{\alpha_k, \kappa} \cap N[G_{P_{\alpha_k}}]$ and $q_{k+1} \leq q_k \restriction [\alpha_k, \kappa)$,’” and
- (3) $p_k \Vdash$ “ $q_{k+1} \Vdash$ ‘ $m_k \in \omega$ and $\eta_k^* \hat{\wedge} g(\eta_k^*)$ extends $\eta_{m_k}^* \hat{\wedge} f(\eta_{m_k}^*)$,’” and
- (4) $p_{k+1} \restriction \alpha_k = p_k$, and
- (5) $p_k \Vdash$ “ $p_{k+1} \restriction [\alpha_k, \alpha_{k+1}) \leq q_{k+1} \restriction \alpha_{k+1}$.’”

The construction proceeds as follows. Given p_k and q_k , work in $V[G_{P_{\alpha_k}}]$ with $p_k \in G_{P_{\alpha_k}}$.

Build $\langle q_k^i : i \in \omega \rangle \in N[G_{P_{\alpha_k}}]$ and $f_k \in \mathcal{B} \cap N[G_{P_{\alpha_k}}]$ such that $q_k^0 = q_k \restriction [\alpha_k, \kappa)$, and for every $i \in \omega$ we have the following:

- (1) $q_k^{i+1} \leq q_k^i$, and
- (2) $q_k^{i+1} \Vdash$ “ $f_k(\eta_i^*) = f(\eta_i^*)$.’”

Using Lemma 5.4, choose $g_k \in \mathcal{B} \cap V$ such that $f_k \leq_{\mathcal{B}} g_k$. We may assume $g_k \in N[G_{P_{\alpha_k}}]$. Hence $g_k \in N$. Hence $g_k \leq_{\mathcal{B}} g$.

By Lemma 5.2 we have $f_k \leq_{\mathcal{B}} g$, so we may choose m_k such that $\eta_k^* \hat{\wedge} g(\eta_k^*)$ extends $\eta_{m_k}^* \hat{\wedge} f_k(\eta_{m_k}^*)$.

Let $q_{k+1} = q_k^{m_k+1}$. We have that q_{k+1} and m_k satisfy (2) and (3).

Using the Proper Iteration Lemma we may choose p_{k+1} satisfying (1), (4), and (5).

This completes the recursive construction.

Let $r \in P_\kappa$ be such that for every $k \in \omega$ we have $r \restriction \alpha_k = p_k$.

Suppose, towards a contradiction, that $r' \leq r$ and $r' \Vdash "f \not\leq_{\mathcal{B}} g."$ Fix a P_κ -name k such that $r' \Vdash "\eta_k^* \hat{\wedge} g(\eta_k^*)$ does not extend $\eta_{m_k}^* \hat{\wedge} f(\eta_{m_k}^*)"$.

By strengthening r' we may assume that k and m_k are integers rather than names.

Because $r' \leq (p_{k+1}, q_{k+1})$ we have $r' \Vdash "\eta_k^* \hat{\wedge} g(\eta_k^*)$ extends $\eta_{m_k}^* \hat{\wedge} f(\eta_{m_k}^*)"$. This is a contradiction.

The Theorem is established.

6 On “the set of reals that are in the ground model has positive outer measure in the forcing extension”

In this section we present a theorem of Shelah ([9, Claim XVIII.3.8B(3)]) that gives a sufficient condition for a forcing iteration to satisfy $\mu^*({}^\omega 2 \cap V) > 0$. This notion has been investigated also by [4].

Definition 6.1. We let \mathcal{B}' be the set of functions f from ω into ${}^{<\omega} 2$ such that $\Sigma\{\mu(U_{f(m)}) : m \in \omega\} \leq 1$.

Lemma 6.2. Suppose $g \in {}^\omega 2$ and λ is a sufficiently large regular cardinal and N is a countable elementary substructure of H_λ . Then g is random over N iff $(\forall f \in \mathcal{B}' \cap N)(\exists m \in \omega)(\forall i \geq m)(g \text{ does not extend } f(i))$.

Proof: We first establish the “only if” direction. Suppose $g \in {}^\omega 2$ and $f \in \mathcal{B}' \cap N$ and $(\exists^\infty m \in \omega)(g \text{ extends } f(m))$. Let $B = \{h \in {}^\omega 2 : (\exists^\infty m \in \omega)(f(m) \text{ is an initial segment of } h)\}$. Then $B \subseteq {}^\omega 2$ is a Borel set and $g \in B \in N$, and $\mu(B) = 0$ because for every $n \in \omega$ we have that B is covered by $\bigcup\{U_{f(i)} : i \geq n\}$, and $\lim_{n \rightarrow \infty} (\mu(\bigcup\{U_{f(i)} : i \geq n\})) = 0$. Therefore g is not random over N .

To prove the “if” direction, suppose that g is not random over N . We may choose $B \in N$ a Borel set of measure zero such that $g \in B \in N$. Let $\langle D_n : n \in \omega \rangle \in N$ be a sequence of open subsets of ${}^\omega 2$ such that for every $n \in \omega$ we have $B \subseteq D_n$ and $\mu(D_n) < 2^{-n}$. For each $n \in \omega$ choose $k_n \leq \omega$ and $\langle \eta_i^n : i < k_n \rangle$ a sequence of pairwise incomparable elements of ${}^{<\omega} 2$ such that $D_n = \bigcup \{U_{\eta_i^n} : i < k_n\}$. Furthermore we may assume that $\langle \langle \eta_i^n : i < k_n \rangle : n \in \omega \rangle$ is an element of N . Let $f \in N$ be a one-to-one function mapping ω onto $\{\eta_i^n : i < k_n \text{ and } n \in \omega\}$. Then we have that $f \in \mathcal{B}'$ and $(\exists^\infty m \in \omega)(g \in U_{f(m)})$. The Lemma is established.

Lemma 6.3. *Suppose $g \in {}^\omega 2$ and λ is a sufficiently large regular cardinal and N is a countable elementary substructure of H_λ . Suppose g is random over N . Suppose $Y \in N$ is a subset of ${}^{<\omega} 2$ and $\Sigma\{\mu(U_\eta) : \eta \in Y\}$ is finite. Then $\{\eta \in Y : g \in U_\eta\}$ is finite.*

Proof: We may assume Y is infinite. Choose a finite integer m and infinite sets (not necessarily disjoint) $D_i \subseteq Y$ for $i < m$ such that each D_i is in N and $\bigcup \{D_i : i < m\} = Y$ and for each $i < m$ we have $\Sigma\{\mu(U_\eta) : \eta \in D_i\} \leq 1$. For each $i < m$ choose $f_i \in N$ such that f_i maps ω onto D_i . By Lemma 6.2, for every $i < m$ there is $\beta_i \in \omega$ such that $(\forall j \geq \beta_i)(g \text{ does not extend } f_i(j))$. Hence $\{\eta \in Y : g \in U_\eta\} \subseteq \bigcup \{\{f_i(j) : j < \beta_i\} : i < m\}$, which is finite. The Lemma is established.

Lemma 6.4. *Suppose P is a poset such that whenever λ is a sufficiently large regular cardinal and N is a countable elementary substructure of H_λ and $P \in N$ and $g \in {}^\omega 2$ and g is random over N , then $V[G_P] \models “g \text{ is random over } N[G_P].”$ Then $V[G_P] \models “{}^\omega 2 \cap V$ has positive outer measure.”*

Proof: Suppose, towards a contradiction, that in $V[G_P]$ we have that B is a Borel subset of ${}^\omega 2$ such that ${}^\omega 2 \cap V \subseteq B$ and $\mu(B) = 0$.

In V , choose λ a sufficiently large regular cardinal and N a countable elementary substructure of H_λ such that $p \in N$ and a name for B is in N . Let $g \in {}^\omega 2$ be random over N . By hypothesis, $V[G_P] \models “g \text{ is random over } N[G_P].”$ Therefore $V[G_P] \models “g \notin B.”$ This contradiction establishes the Lemma.

Lemma 6.5. *Suppose P is a poset. Suppose χ is a sufficiently large regular cardinal and λ is a regular cardinal sufficiently larger than χ . Suppose N is a countable elementary substructure of H_λ and N_1 and N_2 are countable elementary substructures of H_χ and $\chi \in N$ and $P \in N_1 \in N_2 \in N$. Suppose also*

- (1) $G_1 \subseteq P \cap N_1$ is an N_1 -generic subset of P , and
- (2) $p \in G_1$ and $G_1 \in N$, and
- (3) $\langle f_l : l \leq k \rangle \in N$ is a finite sequence of P -names such that $p \Vdash \text{"}f_l \in \mathcal{B}' \cap N_1[G_P]\text{"}$ for all $l \leq k$, and
- (4) $g \in {}^\omega 2$ is random over N , and
- (5) $\langle \beta_l : l \leq k \rangle$ is a sequence of integers and for all $l \leq k$ we have $(\forall j \geq \beta_l)(g \text{ does not extend } f_l[G_1](j))$. That is, for every $j \geq \beta_l$ there is $p' \in G_1$ and $\rho \in {}^{<\omega} 2$ such that g does not extend ρ and $p' \Vdash \text{"}\rho = f_l(j)\text{"}$.

Then there is $G_2 \subseteq P \cap N_2$ an N_2 -generic subset of P such that $p \in G_2$ and $G_2 \in N$ and for all $l \leq k$ we have $(\forall j \geq \beta_l)(g \text{ does not extend } f_l[G_2](j))$.

Proof: Build $\langle p_n : n \in \omega \rangle \in N$ and $\langle m_n : n \in \omega \rangle \in N$ and $\langle f_l^* : l \leq k \rangle \in N$ such that $p_0 = p$ and for each $n \in \omega$ we have each of the following:

- (1) $p_n \in G_1$ and $p_{n+1} \leq p_n$, and
 - (2) m_n is an integer such that $m_n \geq n$ and $p_n \Vdash \text{"}\Sigma\{\mu(U_{f_l(i)}) : i \geq m_n\} < 2^{-n}\text{"}$ for each $l \leq k$," and
 - (3) for every $l \leq k$ we have $f_l^* \in N$ maps ω into ${}^{<\omega} 2$, and
 - (4) $p_n \Vdash \text{"}f_l \restriction m_n = f_l^* \restriction m_n \text{ for each } l \leq k\text{"}$.
- Claim 1. For $l \leq k$ we have $f_l^* \in \mathcal{B}'$.

Proof. Suppose, towards a contradiction, that $l \leq k$ and $m \in \omega$ and $\Sigma\{\mu(U_{f_l^*(i)}) : i < m\} > 1$. Because $p_m \Vdash \text{"}f_l \restriction m = f_l^* \restriction m\text{"}$, it follows that $p_m \Vdash \text{"}f_l \notin \mathcal{B}'\text{"}$. This contradiction establishes the Claim.

Build $\langle p_{n,m} : m \in \omega, n \in \omega \rangle \in N$ and $\langle f_{l,n}^* : l \leq k, n \in \omega \rangle \in N$ such that each of the following holds:

- (1) for every $n \in \omega$ we have that $\langle p_{n,m} : m \in \omega \rangle$ is an N_2 -generic sequence for P and $p_{n,0} = p_n$, and
- (2) for every $l \leq k$ and $n \in \omega$ and $m \in \omega$ we have $p_{n,m} \Vdash \text{"}f_{l,n}^* \restriction m = f_l \restriction m\text{"}$.

Claim 2. For $l \leq k$ and $n \in \omega$ we have $f_{l,n}^* \in \mathcal{B}'$.

Proof: Similar to Claim 1.

Claim 3. For every $l \leq k$ and $n \in \omega$ we have $\Sigma\{\mu(U_{f_{l,n}^*(i)}) : i \geq m_n\} \leq 2^{-n}$.

Proof: Suppose l and n constitute a counterexample. Then we can choose an integer t so large that $\Sigma\{\mu(U_{f_{l,n}^*(i)}) : m_n \leq i < t\} > 2^{-n}$. We have $p_{n,t} \Vdash \text{"}\Sigma\{\mu(U_{f_{l,n}^*(i)}) : m_n \leq i < t\} = \Sigma\{\mu(U_{f_l^*(i)}) : m_n \leq i < t\} < \Sigma\{\mu(U_{f_l^*(i)}) : m_n \leq i < \omega\} < 2^{-n}\text{"}$. This contradiction establishes the Claim.

For each $l \leq k$ and $n \in \omega$ let $U_{l,n}^* = \bigcup \{U_{f_{l,n}^*(i)} : i \in \omega\}$.

Claim 4. For every $l \leq k$ and $n \in \omega$ we have $U_{l,n}^* \subseteq \bigcup \{U_{f_l^*(i)} : i \in \omega\} \cup \bigcup \{U_{f_{l,n}^*(i)} : i \geq m_n\}$.

Proof: The Claim is forced by the condition $p_{n,n}$, hence it is true outright.

For each $l \leq k$ let $U_l^* = \bigcup \{U_{l,n}^* : n \in \omega\}$. By Claims 3 and 4 we have that $\mu(U_l^*)$ is finite for every $l \leq k$. By Lemma 6.4 we have that $\{\rho \in {}^{<\omega}2 : (\exists l \leq k)(\exists n \in \omega)(\exists i \in \omega)(\rho = f_{l,n}^*(i) \text{ and } g \text{ extends } \rho)\}$ is finite. Therefore, we may fix n^* so large that $(\forall l \leq k)(\forall n \in \omega)(\forall i \in \omega)(g \text{ extends } f_{l,n}^*(i) \text{ only if } \mu(U_{f_{l,n}^*(i)}) \geq 2^{-n^*})$.

Claim 5. Suppose $l \leq k$ and $i \in \omega$ and $n \in \omega$ and $\mu(U_{f_{l,n}^*(i)}) \geq 2^{-n^*}$. Then $i < m_{n^*}$.

Proof: Suppose $i \geq m_{n^*}$. Then $p_{n,i+1} \Vdash \mu(U_{f_{l,n}^*(i)}) = \mu(U_{f_l(i)}) < \Sigma\{\mu(U_{f_l(j)}) : j \geq m_{n^*}\} < 2^{-n^*}$. This contradiction establishes the Claim.

Fix $t > m_{n^*}$ such that $t > \beta_l$ for every $l \leq k$. For every $l \leq k$ we have $p_{n^*,t} \Vdash \text{“} f_{l,n^*}^* \restriction t = f_l^* \restriction t \text{”}$. Thus, by Claim 5, we have that $p_{n^*,t} \Vdash \text{“} (\forall l \leq k)(\forall i \geq \beta_l)(g \text{ does not extend } f_{l,n^*}^*(i)) \text{”}$.

Let $G_2 = \{p' \in P \cap N_2 : (\exists m \in \omega)(p_{n^*,m} \leq p')\}$. We have that G_2 is as required.

The Lemma is established.

Definition 6.6. Suppose $g \in {}^\omega 2$. We say that P is g -good iff **whenever**

- (1) χ is a sufficiently large regular cardinal and λ is a regular cardinal sufficiently larger than χ and
- (2) N is a countable elementary substructure of H_λ and $\chi \in N$ and N_1 is a countable elementary substructure of H_χ and
- (3) $P \in N_1 \in N$ and
- (4) g is random over N and
- (5) $k \in \omega$ and $\langle f_l : l < k \rangle \in N$ is a sequence of P -names and
- (6) $p \in P \cap N_1$ and
- (7) $p \Vdash \text{“} (\forall l < k)(f_l \in \mathcal{B}' \cap N_1[G_P]) \text{”}$ and
- (8) $\langle f_l^* : l < k \rangle$ is a sequence of elements of \mathcal{B}' and $\langle \beta_l : l < k \rangle$ is a sequence of integers and for every $l < k$ we have $(\forall m \geq \beta_l)(g \text{ does not extend } f_l^*(m))$ and
- (9) $G_1 \subseteq P \cap N_1$ and $G_1 \in N$ and G_1 is N_1 -generic over P and $p \in G_1$ and

$$(10) (\forall l < k)(f_l[G_1] = f_l^*),$$

then there is $q \leq p$ such that q is N -generic and $q \Vdash$ “ g is random over $N[G_P]$ and $(\forall l < k)(\forall m \geq \beta_l)(g \text{ does not extend } f_l(m))$.”

Lemma 6.7. *Suppose we have that*

- (1) $g \in {}^\omega 2$ and
- (2) $\mathbf{1} \Vdash$ “ Q is g -good,” and
- (3) χ is a sufficiently large regular cardinal and λ is a regular cardinal sufficiently larger than χ , and
- (4) N is a countable elementary substructure of H_λ and $\{P * Q, \chi\} \in N$, and
- (5) $p \in P$ is N -generic and q is a P -name and
- (6) $p \Vdash$ “ N_1 is a countable elementary substructure of $H_\chi[G_P]$ and $N_1 \in N[G_P]$ and g is random over $N[G_P]$ and $q \in Q \cap N_1$,” and
- (7) $k \in \omega$ and $p \Vdash$ “ $\langle f_l : l < k \rangle \in N[G_P]$ is a sequence of Q -names and $q \Vdash_Q$ ‘ $(\forall l < k)(f_l \in \mathcal{B}' \cap N_1[G_Q])$,” and
- (8) $\langle f_l^* : l < k \rangle$ and $\langle \beta_l : l < k \rangle$ are sequences of P -names and $p \Vdash$ “ $(\forall l < k)(f_l^* \in \mathcal{B}' \cap N[G_P])$ and $\beta_l \in \omega$ and $(\forall i \geq \beta_l)(g \text{ does not extend } f_l^*(i))$,” and
- (9) G is a P -name and $p \Vdash$ “ $G \subseteq Q \cap N_1$ is generic over N_1 and $q \in G \in N[G_P]$ and $(\forall l < k)(f_l^* = f_l[G])$.”

Then there is a P -name r such that $p \Vdash$ “ $r \leq q$ ” and (p, r) is N -generic and $(p, r) \Vdash$ “ g is random over $N[G_{P*Q}]$ and $(\forall l < k)(\forall i \geq \beta_l)(g \text{ does not extend } f_l(i))$.”

Proof: Immediate.

Theorem 6.8. *Suppose $g \in {}^\omega 2$ and suppose $\langle P_\eta : \eta \leq \kappa \rangle$ is a countable support forcing iteration based on $\langle Q_\eta : \eta < \kappa \rangle$. Suppose for every $\eta < \kappa$ we have $\mathbf{1} \Vdash_{P_\eta}$ “ Q_η is proper and g -good.” Suppose also*

- (1) χ is a sufficiently large regular cardinal and λ is a regular cardinal sufficiently larger than χ , and
- (2) N is a countable elementary substructure of H_λ and $\{P_\kappa, \chi\} \in N$ and
- (3) $\alpha \in \kappa \cap N$ and $p \in P_\alpha \cap N$ and

- (4) $p \Vdash "N' \text{ is a countable elementary substructure of } H_\chi[G_{P_\alpha}] \text{ and } P_{\alpha,\kappa} \in N' \in N[G_{P_\alpha}] \text{ (so necessarily } \alpha \in N'),"$ and
- (5) p is N -generic and g is a P_α -name and $p \Vdash "g \text{ is random over } N[G_{P_\alpha}] \text{ and } q \in P_{\alpha,\kappa} \cap N',"$ and
- (6) $k \in \omega$ and $p \Vdash "\langle f_l : l < k \rangle \in N[G_{P_\alpha}] \text{ is a sequence of } P_{\alpha,\kappa}\text{-names and } q \Vdash_{P_{\alpha,\kappa}} '(\forall l < k)(f_l \in \mathcal{B}' \cap N'[G_{P_{\alpha,\kappa}}]),'"$ and
- (7) $\langle f_l^* : l < k \rangle$ and $\langle \beta_l : l < k \rangle$ are sequences of P_α -names and $p \Vdash "(\forall l < k)(f_l^* \in \mathcal{B}' \cap N[G_{P_\alpha}] \text{ and } \beta_l \in \omega \text{ and } (\forall i \geq \beta_l)(g \text{ does not extend } f_l^*(i))),"$ and
- (8) G is a P_α -name and $p \Vdash "G \subseteq P_{\alpha,\kappa} \cap N'$ is generic over N' and $q \in G \in N[G_{P_\alpha}]$ and $(\forall l < k)(f_l^* = f_l[G])."$

Then there is $r \in P_\kappa$ such that $r \restriction \alpha = p$ and $p \Vdash "r \restriction [\alpha, \kappa] \leq q"$ and r is N -generic and $r \Vdash "g \text{ is random over } N[G_{P_\kappa}] \text{ and } (\forall l < k)(\forall i \geq \beta_l)(g \text{ does not extend } f_l(i))."$

Proof: By induction on κ .

Successor case: $\kappa = \gamma + 1$.

In $V[G_{P_\alpha}]$ let $G_1 = G \restriction \gamma$ and $G_2 = G/G_1$. That is, $G_1 = \{p' \restriction \gamma : p' \in G\}$ and $(\forall p' \in P_{\alpha,\gamma})(\forall r')(p' \restriction \gamma \Vdash "r' \in G_2" \text{ iff } (\forall p^* \leq p')(\exists q' \in G)(\exists p^\# \leq p^*)(p^\# \leq q' \restriction \gamma \text{ and } p^\# \Vdash "r' = q'(\gamma)"))$.

Choose N^* a countable elementary substructure of $H_\chi[G_{P_\alpha}]$ such that $N'[G_{P_\alpha}] \in N^* \in N[G_{P_\alpha}]$ and $G \in N^*$. Choose $\langle f_l^{**} : l < k \rangle$ such that for all $l < k$ we have $(p, q \restriction \gamma) \Vdash_{P_\gamma} "f_l^{**} = f_l[G_2]"$. Because $p \Vdash "f_l^{**}[G_1] = f_l^*" \text{ for all } l < k$, we have that $(p, q \restriction \gamma) \Vdash "(\forall l < k)(\forall j \geq \beta_l)(g \text{ does not extend } f_l^{**}(j))."$

Use Lemma 6.5 to choose G'_1 such that $p \Vdash "G'_1 \subseteq P_{\alpha,\gamma} \cap N^*" \text{ is generic over } N^* \text{ and } q \restriction \gamma \in G'_1 \text{ and } G'_1 \in N[G_{P_\alpha}] \text{ and } (\forall l < k)(\forall j \geq \beta_l)(g \text{ does not extend } f_l^{**}[G'_1](j))."$

By the induction hypothesis, with G'_1 playing the role of G and $\langle f_l^{**} : l < k \rangle$ playing the role of $\langle f_l : l < k \rangle$, we can choose $r' \in P_\gamma$ such that $r' \restriction \alpha = p$ and $p \Vdash "r' \restriction [\alpha, \gamma] \leq q \restriction \gamma"$ and r' is N -generic and $r' \Vdash "g \text{ is random over } N[G_{P_\gamma}] \text{ and } (\forall l < k)(\forall i \geq \beta_l)(g \text{ does not extend } f_l^{**}(i))."$

Using Lemma 6.7 with G_2 playing the role of G and $N'[G_{P_\gamma}]$ playing the role of N_1 , we may choose r^* such that $r' \restriction \gamma \Vdash "r^* \in Q_\gamma \text{ and } r^* \leq q(\gamma)"$ and $(r', r^*) \Vdash "g \text{ is random over } N[G_{P_\kappa}] \text{ and } (\forall l < k)(\forall i \geq \beta_l)(g \text{ does not extend } f_l(i))."$

Let $r = (r', r^*)$. This concludes the verification of the successor case.

Limit case: κ is a limit ordinal.

Let $\langle \alpha_n : n \in \omega \rangle$ be an increasing sequence from $\kappa \cap N$ cofinal in $\sup(\kappa \cap N)$ such that $\alpha_0 = \alpha$. Let $\langle \sigma_n : n \in \omega \rangle$ list all P_κ -names σ such that $\sigma \in N$ and $\mathbf{1} \Vdash_{P_\kappa} \text{“}\sigma \text{ is an ordinal.”}$ Let $\langle f_l : l \in \omega \rangle$ be a sequence that extends $\langle f_l : l < k \rangle$, such that it lists the set of all P_κ -names f in N such that $(p, q) \Vdash \text{“}f \in \mathcal{B}'\text{.”}$

Build $\langle p_n, q_n, \beta_n, G_n, G_n^*, G'_n, N_n \rangle$ such that $p_0 = p$ and $q_0 = q$ and $G_0 = G$ and $N_0 = N'$ and $\langle \beta_l : l \in \omega \rangle$ extends $\langle \beta_l : l < k \rangle$, and for every $n \in \omega$ we have that each of the following holds:

- (1) $p_n \Vdash \text{“}G'_n = G_n \restriction \alpha_{n+1} \text{ and } G_{n+1} = G_n / G'_n \text{ (see the successor case, above),”}$ and
- (2) $p_n \Vdash \text{“}N_{n+1} \text{ is a countable elementary substructure of } H_\chi[G_{P_{\alpha_n}}] \text{ and } \{N_n[G_{P_{\alpha_{n-1}, \alpha_n}}], G_n, f_n, \alpha_{n+1}, \sigma_n\} \in N_{n+1} \in N[G_{P_{\alpha_n}}] \text{ (if } n = 0 \text{ then replace } N_0[G_{P_{\alpha_{-1}, \alpha_0}}] \text{ with } N_0), \text{ and}$
- (3) β_n is a P_{α_n} -name for an integer and $p_n \Vdash \text{“}(\forall j \geq \beta_n)(g \text{ does not extend } f_n[G_n](j))\text{,”}$ and
- (4) $p_n \Vdash \text{“}G_n^* \subseteq P_{\alpha_n, \alpha_{n+1}} \cap N_{n+1} \text{ is } N_{n+1}\text{-generic and } q_n \restriction \alpha_{n+1} \in G_{n+1}^* \in N[G_{P_{\alpha_n}}] \text{ and } (\forall l < \max(n+1, k))(\forall j \geq \beta_l)(g \text{ does not extend } f_l[G_{n+1}][G_n^*](j))\text{,”}$ and
- (5) $p_{n+1} \in P_{\alpha_{n+1}}$ is N -generic and $p_{n+1} \Vdash \text{“}g \text{ is random over } N[G_{P_{\alpha_{n+1}}}] \text{ and } (\forall i < \max(n+1, k))(\forall j \geq \beta_i)(g \text{ does not extend } f_i[G_{n+1}](j))\text{,”}$ and
- (6) $p_n \Vdash \text{“}p_{n+1} \restriction [\alpha_n, \alpha_{n+1}) \leq q_n \restriction \alpha_{n+1}\text{,”}$ and
- (7) $p_{n+1} \Vdash \text{“}q_{n+1} \leq q_n \restriction [\alpha_{n+1}, \kappa) \text{ and } q_{n+1} \in G_{n+1} \text{ and } q_{n+1} \text{ decides the value of } \sigma_n \text{ and } q_{n+1} \text{ decides the value of } f_l \restriction n \text{ for every } l \leq n\text{.”}$

The construction proceeds as follows. Given p_n and q_n and G_n , construct G'_n and G_{n+1} as in (1) (see successor case, above). There is no problem in choosing N_{n+1} as in (2). We have that $p_n \Vdash \text{“}f_n[G_n] \in \mathcal{B}'\text{”}$ by the reasoning of Claim 1 in the proof of Lemma 6.5, hence we may choose β_n as in (3) because of Lemma 6.2. We may choose G_n^* as in (4) by Lemma 6.5. We may choose p_{n+1} satisfying (5) and (6) by using the induction hypothesis. There is no difficulty in choosing q_{n+1} satisfying (7).

Take $r \in P_\kappa$ such that for every $n \in \omega$ we have $r \restriction \alpha_n = p_n$.

Claim. $r \Vdash \text{“}g \text{ is random over } N[G_{P_\kappa}]\text{.”}$

Proof: Suppose not. By Lemma 6.2 we may take $r' \leq r$ and $l \in \omega$ such that $r' \Vdash \text{“}(\exists^\infty m \in \omega)(g \text{ extends } f_l(m))\text{.”}$ By strengthening r' further, we may assume there is an integer β^* such that $r' \Vdash \text{“}\beta_l = \beta^*\text{.”}$ By a further strengthening of r'

we may assume there is an integer $j \geq \beta^*$ such that $r' \Vdash "g \text{ extends } f_l(j)." \text{ Let } n = \max(j+1, l+1). \text{ By (7) we have that } p_{n+1} \Vdash "q_{n+1} \Vdash 'f_l[G_{n+1}](j) = f_l(j).'" \text{ We have } p_{n+1} \Vdash "g \text{ does not extend } f_l[G_{n+1}](j)." \text{ The Claim is established.}$

We have that r is N -generic by the usual argument on ordinal names in N , and it is clear that $r \Vdash "(\forall l < k)(\forall j \geq \beta_l)(g \text{ does not extend } f_l(j))."$

The Theorem is established.

The following Theorem is [9, Claim XVIII.3.8C(1)].

Theorem 6.9. *Suppose $\langle P_\eta : \eta \leq \kappa \rangle$ is a countable support iteration based on $\langle Q_\eta : \eta < \kappa \rangle$ and for every $\eta < \kappa$ we have $1 \Vdash_{P_\eta} "Q_\eta \text{ is proper and for every } g \in {}^\omega 2 \text{ we have that } Q_\eta \text{ is } g\text{-good}."$ Then $V[G_{P_\kappa}] \models "{}^\omega 2 \cap V \text{ does not have measure zero}."$*

Proof: By Theorem 6.8 with $\alpha = k = 0$ and Lemma 6.2.

7 Preservation of “the set of old reals is non-meager”

Let \mathcal{B}^* be the set of functions from ${}^{<\omega}2$ into ${}^{<\omega}2$.

Definition 7.1. *Suppose $f \in \mathcal{B}^*$ and $g \in {}^\omega 2$. We say $f R^\dagger g$ iff $(\exists^\infty m \in \omega) (g \restriction m \hat{=} f(g \restriction m))$ is an initial segment of g .*

Lemma 7.2. *Suppose $X \subseteq {}^\omega 2$. Then X is non-meager iff for every $f \in \mathcal{B}^*$ there is $g \in X$ such that $f R^\dagger g$.*

Proof: Suppose X is non-meager, and suppose $f \in \mathcal{B}^*$.

For every $i \in \omega$ let $D_i = \bigcup \{U_{\tau \hat{=} f(\tau)} : (\exists n > i)(\tau \in {}^n 2)\}$. We have that each D_i is an open dense set, so because X is non-meager, we may fix $g \in X \cap \bigcap \{D_i : i \in \omega\}$. Clearly $f R^\dagger g$.

For the converse, suppose $(\forall f \in \mathcal{B}^*)(\exists g \in X)(f R^\dagger g)$, and suppose $\langle D_i : i \in \omega \rangle$ is a decreasing sequence of open dense subsets of ${}^\omega 2$. We show $X \cap \bigcap \{D_i : i \in \omega\}$ is non-empty. It suffices to find $g \in X$ such that $(\exists^\infty j \in \omega)(g \in D_j)$.

Choose $f \in \mathcal{B}^*$ such that for every $\eta \in {}^{<\omega}2$ we have $U_{\eta \hat{=} f(\eta)} \subseteq D_{\text{lh}(\eta)}$. Fix $g \in X$ such that $f R^\dagger g$. Given $i \in \omega$ choose $j > i$ such that $g \restriction j \hat{=} f(g \restriction j)$ is an initial segment of g . Let $\eta = g \restriction j$. Then $g \in U_{\eta \hat{=} f(\eta)} \subseteq D_j$.

The Lemma is established.

Lemma 7.3. *Suppose λ is a sufficiently large regular cardinal and N is a countable elementary substructure of H_λ , and suppose $g \in {}^\omega 2$. The following are equivalent:*

- (1) $(\forall f \in \mathcal{B}^* \cap N)(fR^\dagger g)$.
- (2) $(\forall f \in \mathcal{B}^* \cap N)(\exists m \in \omega)(g \restriction m \hat{~} f(g \restriction m))$ is an initial segment of g .
- (3) g is Cohen over N .

Proof: It is obvious that (1) implies (2).

Suppose (2) holds and $D \in N$ is an open dense subset of ${}^\omega 2$. Choose $f \in \mathcal{B}^* \cap N$ such that $(\forall \nu \in {}^{<\omega} 2)(U_{\nu \hat{~} f(\nu)} \subseteq D)$. Using (2), choose $m \in \omega$ such that $g \restriction m \hat{~} f(g \restriction m)$ is an initial segment of g . We have $g \in U_{g \restriction m \hat{~} f(g \restriction m)} \subseteq D$. We conclude that $g \in \bigcap \{D \in N : D \text{ is an open dense subset of } {}^\omega 2\}$, i.e., g is Cohen over N .

Finally, suppose (3) holds and $f \in \mathcal{B}^* \cap N$. Suppose $k \in \omega$. Let $D_k = \{h \in {}^\omega 2 : (\exists m > k)(h \restriction m \hat{~} f(h \restriction m) \text{ is an initial segment of } h)\}$. It is easy to see that for every $k \in \omega$ we have D_k is an open dense subset of ${}^\omega 2$. Because $(\forall k \in \omega)(g \in D_k)$ we have that $fR^\dagger g$.

The Lemma is established.

The following Lemma, due to Goldstern and Shelah, is [9, Lemma XVIII.3.11].

Lemma 7.4. *Suppose P is a Suslin proper forcing (see [1, Section 7]) and for every forcing Q we have $\mathbf{1} \Vdash_Q "P \text{ is Suslin proper and } \mathbf{1} \Vdash_P {}^\omega 2 \cap V[G_Q] \text{ is not meager}."$ Suppose λ is a sufficiently large regular cardinal and N is a countable elementary submodel of H_λ and $P \in N$ and $p \in P \cap N$ and $g \in {}^\omega 2$ is Cohen over N . Then there is $q \leq p$ such that q is N -generic and $q \Vdash "g \text{ is Cohen over } N[G_P]."$*

The proof presented in [9] is quite clear, so we do not repeat it here.

Lemma 7.5. *Suppose $\langle P_\eta : \eta \leq \kappa \rangle$ is a countable support forcing iteration based on $\langle Q_\eta : \eta < \kappa \rangle$. Suppose for every $\eta < \kappa$ we have $\mathbf{1} \Vdash_{P_\eta} "Q_\eta \text{ is a Suslin proper forcing and for every forcing } Q \text{ we have } \mathbf{1} \Vdash_Q "Q_\eta \text{ is Suslin proper and } \mathbf{1} \Vdash_{Q_\eta} {}^\omega 2 \cap V[G_{P_\eta}][G_Q] \text{ is not meager}."$ Suppose λ is a sufficiently large regular cardinal and N is a countable elementary substructure of H_λ and $P_\kappa \in N$ and $\alpha \in \kappa \cap N$ and $p \in P_\alpha$ is N -generic and $p \Vdash "q \in P_{\alpha, \kappa} \cap N[G_{P_\alpha}] \text{ and } g \in {}^\omega 2 \text{ is Cohen over } N[G_{P_\alpha}]."$ Then there is $r \in P_\kappa$ such that r is N -generic and $r \restriction \alpha = p$ and $p \Vdash "r \restriction [\alpha, \kappa] \leq q"$ and $r \Vdash "g \text{ is Cohen over } N[G_{P_\kappa}]."$*

Proof: By induction on κ .

Case 1: κ is a successor ordinal.

Let β be the immediate predecessor of κ . By the induction hypothesis we may take $r' \in P_\beta$ such that r' is N -generic and $r' \restriction \alpha = p$ and $p \Vdash "r' \leq q \restriction \beta"$ and $r' \Vdash "g \text{ is Cohen over } N[G_{P_\beta}]"$. By Lemma 7.4 we may take $r^* \in Q_\beta$ such that $r' \Vdash "r^* \leq q(\beta)"$ and r^* is $N[G_{P_\beta}]$ -generic and $r^* \Vdash "g \text{ is Cohen over } N[G_{P_\beta}][Q_\beta]"$. Let $r \in P_\kappa$ be defined by $r \restriction \beta = r'$ and $r(\beta) = r^*$. We have that r satisfies the requirements of the Lemma.

Case 2: κ is a limit ordinal.

Let $\langle \alpha_k : k \in \omega \rangle$ be an increasing sequence from $\kappa \cap N$ cofinal in $\sup(\kappa \cap N)$ such that $\alpha_0 = \alpha$. Let $\langle \sigma_k : k \in \omega \rangle$ list all P_κ -names σ in N such that $\mathbf{1} \Vdash_{P_\kappa} "\sigma \text{ is an ordinal}"$.

Let $\langle f_i : i \in \omega \rangle$ list all P_κ -names f in N such that $V[G_{P_\kappa}] \models "f \in \mathcal{B}^*",$ and let $\langle \eta'_m : m \in \omega \rangle$ list $^{<\omega}2$.

Build $\langle q_k, p_k, n_k : k \in \omega \rangle$ such that $p_0 = p$ and $q_0 = q$ and for every $k \in \omega$ we have that each of the following holds:

- (1) $p_k \in P_{\alpha_k}$ is N -generic, and
- (2) $p_k \Vdash "q_{k+1} \in P_{\alpha_k, \kappa} \cap N[G_{P_{\alpha_k}}]"$ and $q_{k+1} \leq q_k \restriction [\alpha_k, \kappa),$ and
- (3) $p_k \Vdash "q_{k+1} \Vdash 'g \text{ is Cohen over } N[G_{P_{\alpha_k}}]' \text{ and } \sigma_k \in N \text{ and } n_k \in \omega \text{ and } g \restriction n_k \hat{\wedge} f_k(g \restriction n_k) \text{ is an initial segment of } g,"$ and
- (4) $p_{k+1} \restriction \alpha_k = p_k,$ and
- (5) $p_k \Vdash "p_{k+1} \restriction [\alpha_k, \alpha_{k+1}) \leq q_{k+1} \restriction \alpha_{k+1}."$

The construction proceeds as follows. Given p_k and $q_k,$ work in $V[G_{P_{\alpha_k}}]$ with $p_k \in G_{P_{\alpha_k}}.$

Build $\langle q_k^m : m \in \omega \rangle \in N[G_{P_{\alpha_k}}]$ and $f'_k \in \mathcal{B}^* \cap N[G_{P_{\alpha_k}}]$ such that $\langle q_k^m : m \in \omega \rangle$ is a decreasing sequence of elements of $P_{\alpha_k, \kappa}$ and $q_k^0 \leq q_k \restriction [\alpha_k, \kappa)$ and there is an ordinal τ such that $q_k^0 \Vdash " \tau = \sigma_k,"$ and for every $m \in \omega$ we have that $q_k^m \Vdash "f'_k(\eta'_m) = f_k(\eta'_m)." \text{ Necessarily } \tau \in N[G_{P_{\alpha_k}}] \text{ and therefore, because } p_k \in G_{P_{\alpha_k}} \text{ is } N\text{-generic, we have } \tau \in N. \text{ Because } g \text{ is Cohen over } N[G_{P_{\alpha_k}}] \text{ we may use Lemma 7.3 to take } n_k \text{ such that } g \restriction n_k \hat{\wedge} f'_k(g \restriction n_k) \text{ is an initial segment of } g.$

Let $q_{k+1} = q_k^{n_k+1}.$

Using the induction hypothesis, we may choose p_{k+1} as required.

This completes the recursive construction.

Let $r \in P_\kappa$ be such that for every $k \in \omega$ we have $r \restriction \alpha_k = p_k.$

We have that r is N -generic, because for each $k \in \omega$ we have $p_{k+1} \Vdash "q_{k+1} \Vdash \sigma_k \in N."$

Suppose, towards a contradiction, that $r' \leq r$ and $r' \Vdash "g \text{ is not Cohen over } N[G_{P_\kappa}]"$. Choose $r^* \leq r'$ and $k \in \omega$ such that $r \Vdash "(\forall m \in \omega)(g \restriction m \hat{=} f_k(g \restriction m)) \text{ is not an initial segment of } g."$

Because $r^* \leq (p_{k+1}, q_{k+1})$ we have $r^* \Vdash "g_{n_k} \hat{=} f_k(g \restriction n_k) \text{ is an initial segment of } g."$ This is a contradiction.

The Lemma is established.

The following Theorem is [9, Claim XVIII.Claim 3.10C].

Theorem 7.6. *Suppose $\langle P_\eta : \eta \leq \kappa \rangle$ is a countable support forcing iteration based on $\langle Q_\eta : \eta < \kappa \rangle$. Suppose for every $\eta < \kappa$ we have $\mathbf{1} \Vdash_{P_\eta} "Q_\eta \text{ is Suslin proper forcing and for every forcing } Q \text{ we have } \mathbf{1} \Vdash_Q "Q_\eta \text{ is Suslin proper and } \mathbf{1} \Vdash_{Q_\eta} "\omega 2 \cap V[G_{P_\eta}][G_Q] \text{ is not meager."}"$ Then $\mathbf{1} \Vdash_{P_\kappa} "\omega 2 \cap V \text{ is not meager."}$*

Proof: Suppose, towards a contradiction, that $q \in P_\kappa$ and $q \Vdash "\omega 2 \cap V \text{ is meager}"$. By Lemma 7.3 we may take f a P_κ -name for an element of \mathcal{B}^* such that $q \Vdash "(\forall g \in \omega 2 \cap V)(f R^\dagger g \text{ fails})"$. Let λ be a sufficiently large regular cardinal and let N be a countable elementary substructure of H_λ such that $\{P_\kappa, q, f\} \in N$.

Let $g \in \omega 2$ be Cohen over N .

By Lemma 7.5 with $\alpha = 0$ we may take $r \leq q$ such that $r \Vdash "g \text{ is Cohen over } N[G_{P_\kappa}]"$. By Lemma 7.3 we have $r \Vdash "f R^\dagger g."$ This contradiction establishes the Theorem.

8 References

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